ENERGY AND POTENTIAL الطاقة والجهد

4.1 Energy Expended In Moving a Point Charge in an Electric Field

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy, or do work. If we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative; we do not do the work, the field does

Suppose we wish to move a charge Q a distance *dL* in an electric field E. The force on Q due to the electric field is

$$F_E = QE$$

where the subscript reminds us that this force is due to the field. The component of this force in the direction *d*L which we must overcome is

$$F_{EL} = QE.\mathrm{a}_L$$

where a_L = a unit vector in the direction of dL

The force which we must apply is equal and opposite to the force due to the field

$$F_{apply} = -QE.\mathrm{a}_L$$

and our expenditure of energy is the product of the force and distance. That is,

Differential work done by external source moving Q

$$dW = -QE.\mathrm{a}_L dL == -QE.dL$$

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating

$$W = -Q\int_{inital}^{final} E.dL$$

where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

Example: An electrostatic field is given by E = $(x/2 + 2y)\,\mathbf{a_x} + 2x\,\mathbf{a_y}$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu C$ *(a)* from the origin to (4, 0, 0) m, and *(b)* from (4, 0, 0) m to (4, 2, 0) m?

Solution:

(a) The first path is along x-axis, so that the $\text{dL} = dx\mathbf{a_x}$

$$W = -Q\int_{inital}^{final} E.dL = -(-20\times 10^{-6})\int_0^4 \left(\frac{x}{2}+2y\right).dx = 20\times 10^{-6}\left[\frac{x^2}{4}\right]_0^4 = 80\mu\,\text{J}$$

(b) The second path is in the a_y direction, so that the $\text{dL} = dy\mathbf{a_y}$

$$W = -(-20\times 10^{-6})\int_0^2 2x.dy = 20\times 10^{-6}\times 2\times 4[y]_0^2 = 160\times 10^{-6}\times 2 = 320\mu\ \text{J}$$

Example: Find the work done in moving a point charge Q = 5 μC from the origin to (2 m, π/4, *π/2)* spherical coordinates, in the field $E = 5e^{-r/4}\text{a}_r + \frac{10}{r\sin\theta}\text{a}_\emptyset\ \ (V/m)$?

Solution:

$\text{dL} = dr\mathbf{a_r} + rd\theta\mathbf{a_\theta} + r\ \sin\theta\ d\emptyset\ \mathbf{a_\emptyset}$

$$W = -Q\int_{inital}^{final} E.dL$$

$$= -5\mu\int_{inital}^{final}\left(5e^{-r/4}\text{a}_r + \frac{10}{r\sin\theta}\text{a}_\emptyset\right).(dr\mathbf{a_r} + rd\theta\mathbf{a_\theta} + r\ \sin\theta\ d\emptyset\ \mathbf{a_\emptyset})$$

$$= -5\mu\int_0^2 5e^{-r/4}dr - 5\mu\int_0^{\pi/2}\frac{10}{r\sin\theta}r\ \sin\theta\ d\emptyset$$

$$= -25\mu\,(-4)\left|e^{-r/4}\right|_0^2 - 5\mu\,(10)|\emptyset|_0^{\pi/2}$$

$$= 100\mu\left(e^{-1/2} - e^0\right) - 50\mu\times\frac{\pi}{2} = -117.9\mu\,\text{J}$$

Example: uniform line charge lie along z-axis, determine the work expended in carrying Q from a to b along:

(a) Circular path?

(b) Radial path?

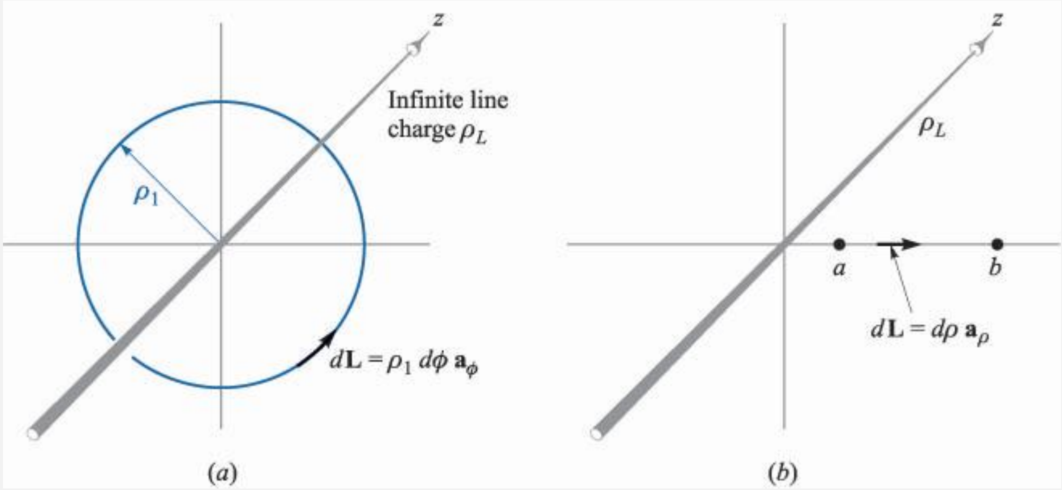


Figure: (a) A circular path and (b) a radial path along which a charge of Q is carried in the field of an infinite line charge.

Solution:

The electric field of a line charge is $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o \rho}\mathbf{a}_\rho$

(a) for circular path $\text{dL} = \rho d\emptyset \mathbf{a}_\emptyset$

$$W = -Q\int_B^A \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho \, .\rho d\emptyset \mathbf{a}_\emptyset = 0 \qquad (\text{a}_\rho . \text{a}_\emptyset) = 0$$

(b) for radial path $\text{dL} = d\rho \mathbf{a}_{\boldsymbol{\rho}}$

$$W = -Q\int_{inital}^{final} \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho \, . d\rho \mathbf{a}_{\boldsymbol{\rho}} = -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon_o}\frac{d\rho}{\rho} = -Q\frac{\rho_L}{2\pi\varepsilon_o}\int_b^a \frac{d\rho}{\rho}$$

$$W = \frac{-Q\rho_L}{2\pi\varepsilon_o}\ln\frac{b}{a}$$

4.2 The Line Integral

A line integral is like many other integrals which appear in advanced analysis, including the surface integral appearing in Gauss's law, in that it is essentially descriptive. It tells us to choose a path, break it up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, and then add the results for all the segments. This is a summation, of course, and the integral is obtained exactly only when the number of segments becomes infinite.

For this special case of uniform electric field intensity, we should note that the work involved in moving the charge depends only on Q, E, and L_{BA}, a vector drawn from the initial to the final point of the path chosen. ***It does not depend on the particular path*** we have selected along which to carry the charge. We may proceed from B to A on a straight line or via the Old Chisholm Trail; the answer is the same.

$$\mathrm{dL} = dx\mathbf{a}_{\boldsymbol{x}} + dy\mathbf{a}_{\boldsymbol{y}} + dz\mathbf{a}_{\boldsymbol{z}}$$

$$\mathrm{dL} = d\rho\mathbf{a}_{\boldsymbol{\rho}} + \rho d\emptyset\mathbf{a}_{\emptyset} + dz\mathbf{a}_{\boldsymbol{z}}$$

$$\mathrm{dL} = dr\mathbf{a}_{\boldsymbol{r}} + rd\theta\mathbf{a}_{\boldsymbol{\theta}} + r\sin\theta\, d\emptyset\, \mathbf{a}_{\emptyset}$$

Example: Given the nonuniform field $E = y\mathrm{a}_x + x\mathrm{a}_y + 2\mathrm{a}_z$, determine the work expended in carrying 2C from B(l, 0, 1) to A(0.8, 0.6,1) along:

(a) The shorter arc of the circle $x^2 + y^2 = 1 \quad z = 1$?

(b) The straight line path from B to A?

Solution:

(a) $W = -Q\int_B^A E.dL = 2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z).(dx\boldsymbol{a_x} + dy\boldsymbol{a_y} + dz\boldsymbol{a_z})$

$$= -2\int_1^{0.8} ydx \; - 2\int_0^{0.6} xdy - 4\int_1^1 dz = -2\int_1^{0.8}\sqrt{1-x^2}dx \; - 2\int_0^{0.6}\sqrt{1-y^2}dy - 0$$

$$= -\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1}y\right]_0^{0.6}$$

$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) = \; -0.96 \text{ J}$$

(b) We start by determining the equations of the straight line

$$y - y_0 = m(x - x_0)$$

$$m = slope = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A} = \frac{0 - 0.6}{1 - 0.8} = \frac{-0.6}{0.2} = -3$$

At point B

$$y - 0 = -3(x - 1)\,,$$

$$y = -3(x - 1)$$

$$W = -2\int_1^{0.8} y dx \; - 2\int_0^{0.6} x dy - 4\int_1^1 dz$$

$$= 6\int_1^{0.8} (x - 1) dx \; - 2\int_0^{0.6} \left(1 - \frac{y}{3}\right) dy = -0.96 \text{ J}$$

4.3 Definition of Potential Difference and Potential

We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field E,

$$W = -Q\int_{inital}^{final} E.dL$$

In much the same way as we defined the electric field intensity as the force on a *unit* test charge, we now define ***potential difference*** *V* as the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$\text{potential difference } = V = -\int_{inital}^{final} E.dL$$

Potential difference is measured in joules per coulomb, for which the *volt* is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_B^A E.dL$$

V_{AB} is positive if work is done in carrying the positive charge from B to A.

4.4 The Potential Field of a Point Charge

The potential difference between points located at r = r_A and r = r_B in the field of a point charge Q placed at the origin

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$

and dL = $dr\mathbf{a_r}$

$$V_{AB} = -\int_B^A E.dL = -\int_{rB}^{rA} \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r\,.dr\mathbf{a_r} = -\int_{rB}^{rA} \frac{Q}{4\pi\varepsilon_o r^2}dr$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{rA} - \frac{1}{rB}\right) = V_A - V_B$$

The potential difference between two points in the field of a point charge depends only on the distance of each point from the charge and does not depend on the particular path used to carry our unit charge from one point to the other

How might we conveniently define a zero reference for potential? The simplest possibility is to let V = 0 at infinity. If we let the point at r = r_B recede to infinity the potential at rA becomes

$$V_A = \frac{Q}{4\pi\varepsilon_o r_A} \qquad \left(\frac{1}{r_B} = \frac{1}{\infty} = 0\right)$$

Since there is no reason to identify this point with the A subscript,

$$V = \frac{Q}{4\pi\varepsilon_o r}$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference.

A convenient method to express the potential without selecting a specific zero reference entails identifying r_A as r once again and letting $Q/4\pi\varepsilon_o r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\varepsilon_o r} + C_1$$

C_1 may be selected so that V = 0 at any desired value of r. We could also select the zero reference indirectly by electing to let V be V_o at r = r_o.

Example: Find the potential at r_A = 5 m with respect to r_B =*15* m due to a point charge Q =500 pC at the origin and zero reference at infinity?

Solution:

$$V_{AB} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{rA} - \frac{1}{rB}\right)$$

$$V_{AB} = \frac{500\times10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{5} - \frac{1}{15}\right) = 0.6\ V$$

The zero reference at infinite must be used to find V_5 and V_{15}

$$V_5 = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r5}\right) = \frac{500\times10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{5}\right) = 0.9\ V$$

$$V_{15} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r15}\right) = \frac{500\times10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{15}\right) = 0.3\ V$$

Example: A 15-nC point charge is at the origin in free space. Calculate V_1 if point P_1 is located at P_1 (-2, 3, -1) and: *(a)* V = 0 at (6, 5, 4); (b) V =0 at infinity; (c) V = 5 V at (2, 0, 4)?

Solution:

a-

$$V_{p1} = \frac{Q}{4\pi\varepsilon_o R_{p1}} + c$$

$$C = V_{ref} - \frac{Q}{4\pi\varepsilon_o R_{ref}}$$

$$C = 0 - \frac{15\times10^{-9}}{4\pi\varepsilon_o\sqrt{6^2+5^2+4^2}} = -15.37$$

$$\therefore V_{p1} = \frac{15\times10^{-9}}{4\pi\varepsilon_o\sqrt{2^2+3^2+1^2}} - 15.37$$

$$V_{p1} = 20.7\ V$$

b- *V=0 at infinity*

$$V_{p1} = \frac{Q}{4\pi\varepsilon_o R_{p1}} = \frac{15 \times 10^{-9}}{4\pi\varepsilon_o\sqrt{2^2+3^2+1^2}} = 36.04\ V$$

c-

$$C = V_{ref} - \frac{Q}{4\pi\varepsilon_o R_{ref}} = 5 - \frac{15 \times 10^{-9}}{4\pi\varepsilon_o\sqrt{2^2+0^2+4^2}} = -25.15$$

$$V_{p1} = 36.04 - 25.15 = 10.89\ V$$

4.5 The Potential Field of a Line Charge

The potential difference between points located at $\rho = a$ and $\rho = b$ in the field of a point charge Q placed at the origin

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$$

and dL $= d\rho\mathbf{a}_{\boldsymbol{\rho}}$

$$V_{AB} = -\int_B^A E.dL = -\int_b^a \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho\,.d\rho\mathbf{a}_{\boldsymbol{\rho}} = -\int_b^a \frac{\rho_L}{2\pi\varepsilon_o\rho}d\rho$$

$$V_{AB} = \frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{b}{a}$$

Example: For a line charge $\rho_L = 10^{-9}/2$ C/m on the z-axis, find V_{AB}, where *A* is (2m, π/2, 0) and *B* is (4m, π, 5m) ?

Solution:

$$V_{AB} = \frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{b}{a}$$

$$V_{AB} = \frac{0.5 \times 10^{-9}}{2\pi\varepsilon_o}\ln\frac{4}{2} = 6.24\ V$$

4.6 The Potential Field of a System of Charges

The potential field of a single point charge, which we shall identify as Q_1 and locate at r1, involves only the distance |r - r1| from Q1 to the point at r. For a zero reference at infinity, we have

$$V(r) = \frac{Q_1}{4\pi\varepsilon_o|r - r_1|}$$

The potential due to two charges, Q_1 at r_1 and Q_2 at r_2, is a function only of $|r - r_1|$ and $|r - r_2|$, the distances from Q_1 and Q_2 to the field point, respectively.

$$V(r) = V(r) = \frac{Q_1}{4\pi\varepsilon_o|r - r_1|} + \frac{Q_2}{4\pi\varepsilon_o|r - r_2|}$$

If the charge distribution takes the form of a line charge, a surface charge, a volume charge the integration is along the line or over the surface or volume:

$$V = \int \frac{\rho_L dL}{4\pi\varepsilon_o|R|}$$

$$V = \int_S \frac{\rho_s ds}{4\pi\varepsilon_o|R|}$$

$$V = \int_{vol} \frac{\rho_v dv}{4\pi\varepsilon_o|R|}$$

Example: Five equal point charges, Q=2 nC, are located at x=2, 3, 4, 5, 6 m. Find the potential at the origin?

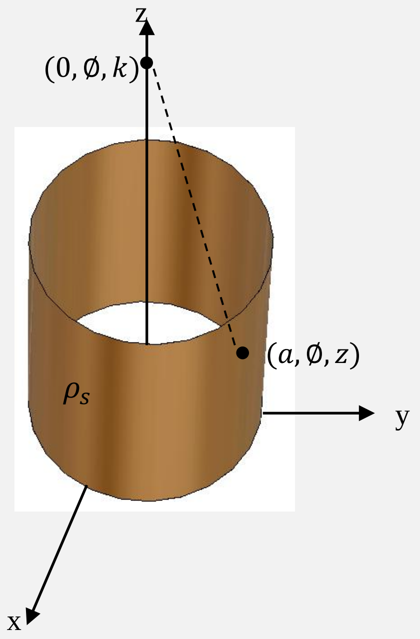
Solution:

$$V = V_1 + V_2 + V_3 + V_4 + V_5$$

$$V_1 = \frac{Q}{4\pi\varepsilon_o r_1} \quad , \quad V_2 = \frac{Q}{4\pi\varepsilon_o r_2}$$

$$\therefore\ V = \frac{2\times 10^{-9}}{4\pi\varepsilon_o}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right) = 261\ V$$

Example: Find the potential V at (0, 0, K) for cylindrical surface charge $\rho_s = \rho_{s0}$, $0 \le z \le h$ and $\rho = a$?



Solution:

$$V = \int_S \frac{\rho_s ds}{4\pi\varepsilon_o |R|}$$

$$ds = \rho\, d\emptyset\, dz = a\, d\emptyset\, dz$$

$$R = -a\, \mathrm{a}_\rho + (k - z)\mathrm{a}_z$$

$$|R| = \sqrt{a^2 + (k - z)^2}$$

$$V = \int_0^h \int_0^{2\pi} \frac{\rho_{so} a\, d\emptyset\, dz}{4\pi\varepsilon_o \sqrt{a^2 + (k - z)^2}}$$

$$V = \frac{\rho_{so} a}{4\pi\varepsilon_o} \times 2\pi \int_0^h \frac{dz}{\sqrt{a^2 + (k - z)^2}}$$

$$V = \frac{-a\rho_{so}}{2\varepsilon_o} \int_0^h \frac{-\, dz}{a\sqrt{1 + \left(\frac{k - z}{a}\right)^2}}$$

$$V = \frac{-\rho_{so}}{2\varepsilon_o} \left[\sinh^{-1}\left(\frac{k - z}{a}\right)\right]_0^h$$

$$V = \frac{\rho_{so}}{2\varepsilon_o} \left[\sinh^{-1}\left(\frac{k - h}{a}\right) - \sinh^{-1}\left(\frac{k}{a}\right)\right]$$

$$* \sinh^{-1} u = \ln\left(u + \sqrt{u^2 + 1}\right)$$

$$V = \frac{\rho_{so}}{2\varepsilon_o}\left[\ln\left(\left(\frac{k - h}{a}\right) + \sqrt{\left(\frac{k - h}{a}\right)^2 + 1}\right) - \ln\left(\left(\frac{k}{a}\right) + \sqrt{\left(\frac{k}{a}\right)^2 + 1}\right)\right]$$

$$\therefore\ V = \frac{\rho_{so}}{2\varepsilon_o} \ln\left(\frac{k + \sqrt{a^2 + k^2}}{k - h + \sqrt{a^2 + (k - h)^2}}\right)$$

4.7 Gradient

The vector field ∇V (also written grad **V)** is called the *gradient* of the scalar function **V**

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(Cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

4.8 Relationship Between E and V

The electric field intensity E may be obtained when the potential function V is known by simply taking the negative of the gradient of V. The gradient was found to be a vector normal to the equipotential surfaces, directed to a positive change in V. With the negative sign here, the E field is found to be directed from higher to lower levels of potential V

$$E = -\nabla V$$

Example: Given the potential field, $V = 2x^2 y - 5z$, and a point $P(-4, 3, 6)$, find at point P: the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v?

Solution:

The potential at P is:

$$V = 2(-4)^2(3) - 5(6) = 66\,V$$

The electric field intensity E is :

$$E = -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

$$E = \frac{-\partial}{\partial x}(2x2\,y\; - \;5z)\mathbf{a}_x + \frac{-\partial}{\partial y}(2x2\,y\; - \;5z)\mathbf{a}_y + \frac{-\partial}{\partial z}(2x2\,y\; - \;5z)\mathbf{a}_z$$

$$E = -4xy\,\mathbf{a}_x - 2x^2\,\mathbf{a}_y + 5\mathbf{a}_z$$

E at the point P is:

$$E = 48\,\mathbf{a}_x - 32\,\mathbf{a}_y + 5\mathbf{a}_z$$

$$D = \varepsilon_o E = \varepsilon_o\left(-4xy\,\mathbf{a}_x - 2x^2\,\mathbf{a}_y + 5\mathbf{a}_z\right)$$

$$\rho_v = \nabla . D = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} = -4\varepsilon_o\,y$$

$$\rho_{v(at\ point\ P)} = -4\varepsilon_o(3) = -106.2\;PC/m^3$$

Example: Given the potential field in cylindrical coordinates $V = \frac{100}{z^2+1}\,\rho\cos\emptyset$, and point P $\rho = 3$ m, $\emptyset = 60^o$, $z=2$ m, in free space find at point P: the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v?

Solution:

The potential field V at the point P is:

$$V = \frac{100}{2^2+1}\,(3)\cos(60) = 30\;V$$

$$E = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\mathbf{a}_\emptyset + \frac{\partial V}{\partial z}\,\mathbf{a}_z\right)$$

$$E = \left(\frac{\partial}{\partial \rho}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \emptyset}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\mathbf{a}_\emptyset + \frac{\partial}{\partial z}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\,\mathbf{a}_z\right)$$

$$E = \frac{-100}{z^2+1}\,\cos\emptyset\;\mathbf{a}_\rho + \frac{-100}{z^2+1}\,\sin\emptyset\;\mathbf{a}_\emptyset + \frac{-200\,z}{(z^2+1)^2}\,\rho\cos\emptyset\;\mathbf{a}_z$$

$$E\ at\ p\ = \frac{-100}{2^2+1}\cos 60\ \mathbf{a}_\rho + \frac{-100}{2^2+1}\sin 60\ \mathbf{a}_\emptyset + \frac{-200(2)}{(2^2+1)^2}\ 3\cos 60\ \mathbf{a}_z$$

$$E\ at\ P\ = -10\ \mathbf{a}_\rho - 17.3\ \mathbf{a}_\emptyset + 24\ \mathbf{a}_z$$

$$D = \varepsilon_o E = \varepsilon_o\left(\frac{-100}{z^2+1}\cos\emptyset\ \mathbf{a}_\rho + \frac{-100}{z^2+1}\sin\emptyset\ \mathbf{a}_\emptyset + \frac{-200\,z}{(z^2+1)^2}\ \rho\cos\emptyset\ \mathbf{a}_z\right)$$

$$\rho_v = \nabla . D = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial\emptyset} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \frac{1}{\rho}\frac{\partial}{\partial\emptyset}\left(\frac{-100}{z^2+1}\sin\emptyset\right) + \frac{\partial}{\partial z}\left(\frac{-200\,z}{(z^2+1)^2}\ \rho\cos\emptyset\right)$$

$$\rho_v = \frac{1}{\rho}\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \frac{1}{\rho}\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \left(\frac{(z^2+1)^2\times(-200) - 200\,z\times 4z(z^2+1)}{(z^2+1)^4}\ \rho\cos\emptyset\right)$$

4.9 The Electric Dipole

An *electric dipole*, or simply a ***dipole***, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to the distance to the point *P* at which we want to know the electric and potential fields.

The dipole is shown in Figure 4.9a. The distant point *P* is described by the spherical coordinate's r, θ, and $\emptyset = 90^o$, in view of the azimuthal symmetry. The positive and negative point charges have separation *d* and rectangular coordinates (0, 0, ½ *d*) and (0, 0, -½ *d*), respectively.

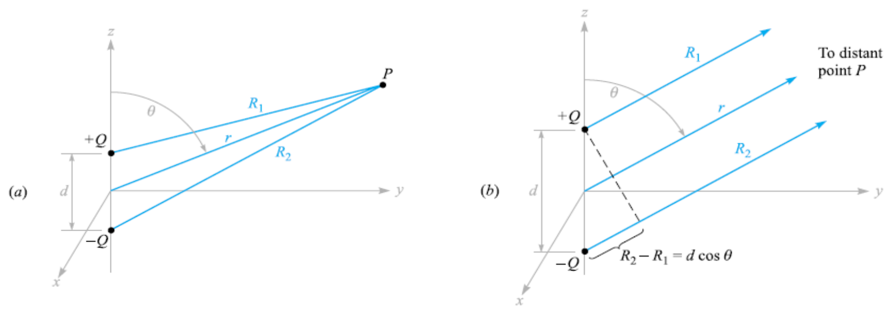


Figure 4.9: The geometry of an electric dipole

The distances from Q and −Q to P be R_1 and R_2, respectively, and write the total potential as

$$V = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_o}\frac{R_2 - R_1}{R_1 R_2}$$

$$R_2 - R_1 = d\ \cos\theta$$

The final result is then

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_o r^2}$$

Using the gradient relationship in spherical coordinates

$$E = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\emptyset}\mathbf{a}_\emptyset\right)$$

$$E = -\left(-\frac{Qd\,cos\,\theta}{2\pi\varepsilon_o r^3}\mathbf{a}_r - \frac{Qd\sin\theta}{4\pi\varepsilon_o r^3}\mathbf{a}_\theta\right)$$

$$E = \frac{Qd}{4\pi\varepsilon_o r^3}\left(2\,cos\,\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right)$$

The potential field of the dipole may be simplified by making use of the ***dipole moment***. We first identify the vector length directed from −*Q* to +*Q* as d and then define the *dipole moment* as *Q*d and assign it the symbol p. Thus

$$\mathrm{P} = Q\mathrm{d}$$

Because $\mathrm{P}.\,\mathrm{a}_r = d\ \cos\theta$, we then have

$$V = \frac{\mathrm{P}.\,\mathrm{a}_r}{4\pi\varepsilon_o r^2}$$

This result may be generalized as

$$V = \frac{\mathrm{P}.\,(r - \grave{r})}{4\pi\varepsilon_o |r - \grave{r}|^3}$$

where *r* locates the field point *P*, and $\grave{r}$ determines the dipole center

Example: A dipole 1nC at (0 , 0 , 0.1) and -1nC at (0, 0 , -0.1) . (*a*) Calculate V *and E* at (0.3, 0, 0.4)?

Solution:

$$V = \frac{Q\ d\ \cos\theta}{4\pi\varepsilon_o r^2}$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{0.3^2 + 0.4^2} = 0.5$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{0.3}{0.4} = 36.9^0$$

$$V = \frac{1\ \times 10^{-9} \times 0.2\ \cos 36.9}{4\pi\varepsilon_o 0.5^2} = 5.76\, V$$

$$E = \frac{Qd}{4\pi\varepsilon_o r^3}(2\, cos\,\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$$

$$E = \frac{1\ \times 10^{-9} \times 0.2}{4\pi\varepsilon_o 0.5^3}(2\, cos\, 36.9\, \mathbf{a}_r + \sin 36.9\, \mathbf{a}_\theta)$$

Example: An electric dipole located at the origin in free space has a moment p = 3$\mathbf{a}_x$ − 2$\mathbf{a}_y$ + $\mathbf{a}_z$ nC.m , find *V* at P_A(2, 3, 4)?

Solution:

$$V = \frac{\mathrm{P}.(r - \grave{r})}{4\pi\varepsilon_o |r - \grave{r}|^3}$$

$$r = 2\mathrm{a}_x + 3\mathrm{a}_y + 4\mathrm{a}_z$$

$$\grave{r} = 0$$

$$V = \frac{(3\mathrm{a}_x - 2\mathrm{a}_y + \mathrm{a}_z).(2\mathrm{a}_x + 3\mathrm{a}_y + 4\mathrm{a}_z - 0)}{4\pi\varepsilon_o |2\mathrm{a}_x + 3\mathrm{a}_y + 4\mathrm{a}_z - 0|^3} = \frac{3 * 2 - 2 * 3 + 1 * 4}{4\pi\varepsilon_o\left(\sqrt{2^2 + 3^2 + 4^2}\right)^3} = 0.23\, V$$

Example: A dipole of moment p = 6$\mathbf{a}_z$ nC.m is located at the origin in free space. (*a*) Find *V* at *P*(r = 4, θ = 20◦,$\emptyset$ = 0◦). (*b*) Find **E** at *P* ?

Solution:

$$V = \frac{P\ \cos\theta}{4\pi\varepsilon_o r^2} = V = \frac{6\times10^{-9}\ \cos 20}{4\pi\varepsilon_o (4)^2} = \ 3.17\,V$$

$$E = \frac{Qd}{4\pi\varepsilon_o r^3}(2\,cos\,\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) = \frac{6\times10^{-9}}{4\pi\varepsilon_o (4)^3}(2\,cos\,20\,\mathbf{a}_r + \sin 20\,\mathbf{a}_\theta)$$

$$E = 1.58\,\mathbf{a}_r + 0.29\,\mathbf{a}_\theta \quad V/m$$

Home work

Q₁: Find *V* on the *z* axis for a uniform line charge ρ_L in the form of a ring, ρ = *a*, in the *z* = 0 plane?

Q_2: Let a uniform ***surface*** charge density of 5 nC/m^2 be present at the *z* = 0 plane, a uniform ***line*** charge density of 8 nC/m be located at *x* = 0, *z* = 4, and a ***point*** charge of 2 μC be present at *P*(2, 0, 0). If *V* = 0 at (0, 0, 5), find *V* at (1, 2, 3)? ***Ans:*** 1.98 KV

Q_3: In spherical coordinates,$E = \frac{2r}{(r^2+a^2)^2}\,\mathrm{a}_r$ V/m. Find the potential at any point, using the reference (*a*)*V* = 0 at infinity; (*b*) *V* = 0 at *r* = 0; (*c*)*V* = 100 V at *r* = *a*.?

Q_4: *Given $V = 80\rho^{0.6}$ V, assuming free space conditions, find. (a) E; (b) the volume charge density at ρ= 0.5 m; (c) the total charge lying within the closed surface ρ= 0.6, 0 < z < 1.*

***Ans**: $-48\rho^{0.4}$, -673 C/m^3 , -1.96 nC*

Q_5: *A dipole having a moment **p** = 3$\mathbf{a}_x$ −5$\mathbf{a}_y$ + 10$\mathbf{a}_z$ nC **.** m is located at Q(1, 2,-4) in free space. Find V at P(2, 3, 4)?* ***Ans: V=1.31***

Q_6: The dipole lie on z-axis and located at the origin, find the difference in potential between points at θ_a and θ_b, each point having the same r and $\emptyset$ coordinates?